POLYA PROBLEM-SOLVING SEMINAR: THE MASTERCLASS

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1. Let n be a positive integer. Let $\mathcal{F}$ be a family of sets that contains more than half of all subsets of an n-element set X. Prove that from $\mathcal{F}$ we can select $\lceil \log_2 n \rceil + 1$ sets that form a separating family of X, i.e., for any two distinct elements of X there is a selected set containing exactly one of the two elements. (Gyujin Oh, Miklos Schweitzer competition 2014)

2. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line l going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line l going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times. (Lisa Sauermann, International Mathematical Olympiad 2011 problem 2)

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded. (Morris Ang, IMO Shortlist 2012)

4. Find all completely multiplicative functions $f : \mathbb{Z} \to \mathbb{N}$ such that for all a and b in $\mathbb{Z}$ with $b \neq 0$, there exist q and r in $\mathbb{Z}$ such that $a = bq + r$ and $f(r) < f(b)$. (In other words, find all "norms" which make $\mathbb{Z}$ into a Euclidean Domain). (Zeb Brady, Miklos Schweitzer competition)

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