POLYA PROBLEM-SOLVING SEMINAR: THE MASTERCLASS

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1. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded. (Morris Ang, IMO Shortlist 2012)

2. For an integer m, define $p(m)$ to be the greatest prime divisor of m, with the exception of the special cases $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f having integer coefficients such that the sequence $\{p(f(n^2)) - 2n\}_{n \geq 0}$ is bounded above. (Alex Zhai, USA Mathematical Olympiad 2006 # 3)

3. Recall that the Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Show that

$$\sum_{n=2}^{\infty} \arctan\left(\frac{(-1)^n}{F_{2n}}\right) = \frac{1}{2} \arctan \frac{1}{2}.$$

(Kevin Li, Problem 4 from the week 1 Polya Seminar)

4. Find all completely multiplicative functions $f : \mathbb{Z} \to \mathbb{N}$ ($\mathbb{N}$ are the nonnegative integers) such that for all a and b in $\mathbb{Z}$ with $b \neq 0$, there exist q and r in $\mathbb{Z}$ such that $a = bq + r$ and $f(r) < f(b)$. (In other words, find all "norms" which make $\mathbb{Z}$ into a Euclidean Domain.) (Zeb Brady, Miklos Schweitzer competition)

Willing to present an enlightening problem? Let us know!

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