# POLYA PROBLEM-SOLVING SEMINAR: THE MASTERCLASS

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**1.** Recall that the Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Show that

$$\sum_{n=2}^{\infty} \arctan\left(\frac{(-1)^n}{F_{2n}}\right) = \frac{1}{2}\arctan\frac{1}{2}.$$

(Kevin Li, Problem 4 from the week 1 Polya Seminar)

**2.** Find all completely multiplicative functions $f : \mathbb{Z} \to \mathbb{N}$ ($\mathbb{N}$ are the nonnegative integers) such that for all $a$ and $b$ in $\mathbb{Z}$ with $b \neq 0$, there exist $q$ and $r$ in $\mathbb{Z}$ such that $a = bq + r$ and $f(r) < f(b)$. (In other words, find all "norms" which make $\mathbb{Z}$ into a Euclidean Domain.) (Zeb Brady, Miklos Schweitzer competition)

**3.** Suppose $p$ is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

(Putnam 1991B4, from the week 3 Polya seminar)

Willing to present an enlightening problem? Let us know!

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*Date*: Monday, October 19, 2015.