POLYA PROBLEM-SOLVING SEMINAR: THE MASTERCLASS

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1. Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

(Putnam 1991B4, from the week 3 Polya seminar)

2. *(Continuation of last week)* Find all completely multiplicative functions $f : \mathbb{Z} \to \mathbb{N}$ ($\mathbb{N}$ are the nonnegative integers) such that for all a and b in $\mathbb{Z}$ with $b \neq 0$, there exist q and r in $\mathbb{Z}$ such that $a = bq + r$ and $f(r) < f(b)$. (In other words, find all "norms" which make $\mathbb{Z}$ into a Euclidean Domain.) (Zeb Brady, Miklos Schweitzer competition)

3. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$, compute

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2}.$$

(Boris Perkhounkov)

Coming soon (next week?): Zhivko Zhechev and Nina Zubrillina.

Also, can anyone solve these problems from last week?

12. Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

15. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

Willing to present an enlightening problem? Let us know!

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