POLYA PROBLEM-SOLVING SEMINAR: THE MASTERCLASS

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– polyaseminar.wordpress.com –

1. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$, compute

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2}.$$

(Boris Perkhounkov, from the Stanford Math Tournament, and created by him)

2. The triangle ABC ($AB > BC$) inscribed in a circle Ω. Points M and N are chosen on sides AB and BC, respectively, so that $AM = CN$. Lines MN and AC intersect at a point K. Let P be the center of the inscribed circle of triangle AMK, and Q the center of the excircle of triangle CNK tangent to side CN. Prove that the midpoint of the arc of the circle ABC of Ω is equidistant from the points P and Q. (Nina Zubrillina)

3. Does there exist a strictly increasing $f : \mathbb{N} \to \mathbb{N}$ which satisfies the following two conditions:

(1) $f(1) = 2$
(2) $f(f(n)) = f(n) + n$ for all $n \in \mathbb{N}$.

Here $\mathbb{N} = \{1, 2, 3, ...\}$. (Zhivko Zhechev, from the 1993 IMO)

4 Let $p(t)$ be a polynomial with all real roots with the condition $p(0) > 0$. Prove that for any positive odd integer m, if we let $f(t) = p(t)^{-m}$, then

$$\sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} x^k > 0$$

for all real numbers x. (Gyujin Oh, from the 1990 Miklos Schweitzer competition)

Willing to present an enlightening problem? Let us know!

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