POLYA PROBLEM-SOLVING SEMINAR: THE MASTERCLASS

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– polyaseminar.wordpress.com –

1. Let $p(t)$ be a polynomial with all real roots with the condition $p(0) > 0$. Prove that for any positive odd integer m, if we let $f(t) = p(t)^{-m}$, then

$$\sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} x^k > 0$$

for all real numbers x. (Gyujin Oh, from the 1990 Miklos Schweitzer competition)

2. Let S_n be the group of all permutations of length n, where $n \geq 3$ and G is a subgroup of S_n generated by $n-2$ transpositions. Show that for any $i \in \{1, \ldots, n\}$, the set $H_i = \{\sigma(i) : \sigma \in G\}$ has at most $n-1$ elements. (Andreea Georgescu)

3. a) 100 people from 50 countries, 2 from each country, are standing in a big circle. Show that we can divide them into two groups such that no group contains two people from the same country or three consecutive people in the circle.
b) 100 people from 25 countries, 4 from each country, are standing in a big circle. Show that we can divide them into four groups such that no group contains two people from the same country or two neighboring people in the circle. (Lisa Sauermann, from the 2005 Russian Olympiad)

4. You and a friend are playing a game against a devil. While your friend is in another room, the devil presents you with a standard (8x8) chessboard with a coin on each square, randomly facing up or down. The devil picks a square and tells you that this square is magic. You then get to choose exactly one coin on the board to turn over. After you flip the coin, your friend is called in and tries to guess which square is the magic square. Can you and your friend devise a strategy beforehand to win this game? You are not allowed to communicate once the game starts. (from Brandon Azad — This is not yet officially proposed as a Masterclass Problem, but it seems too good not to mention!)

Willing to present an enlightening problem? Let us know!

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