POLYA PROBLEM-SOLVING SEMINAR WEEK 2: NUMBER THEORY

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

The Problems.

1. Show that the sum of consecutive primes is never twice a prime.

2. Prove that $2^{70} + 3^{70}$ is divisible by 13. (Larson p. 99)

3. Find the smallest number with 28 divisors. (Larson p. 104)

4. Find the last two digits of $7^{7^{\cdot^{\cdot^{7}}}}$ where the tower contains seven 7's.

5. Several positive integers are written on a chalk board. One can choose two of them, erase them, and replace them with their greatest common divisor and least common multiple. Prove that eventually the numbers on the board do not change.

6. Show that

$$n = \sum_{d|n} \phi(d)$$

where ϕ is Euler's function.

7. Consider two lists. List A consists of the positive powers of 10 (10, 100, 1000, $\ldots$) written in base 2. List B consists of the positive powers of 10 written in base 5. Show that, for any integer $n > 1$, there is exactly one number in exactly one of the lists that is exactly n digits long.

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Powers of 10	List A	List B
10	1010 (4 digits)	20 (2 digits)
100	1100100 (7 digits)	400 (3 digits)
1000	1111101000 (10 digits)	13000 (5 digits)
10000	10011100010000 (14 digits)	310000 (6 digits)

(1994 Asian Pacific Mathematical Olympiad)

8. Show that

$$\sum_{i=1}^{n} \phi(i) \left\lfloor \frac{n}{i} \right\rfloor = \frac{n(n+1)}{2}.$$

9. Call a number n a pseudo-prime (to base 2) if $2^{n-1} \equiv 1 \pmod{n}$. (Note that all odd primes are pseudo-primes.) Show that if p is a prime, then $2^p - 1$ is a pseudo-prime.

10. Show that

$$1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

is not an integer for any $n > 1$.

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