POLYA PROBLEM-SOLVING SEMINAR WEEK 4: INVARIANTS

K. SOUNDARARAJAN AND RAVI VAKIL

The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

The problems.

W1. (a) Can you tile a chessboard (8×8) with 32 dominoes (2×1) ? (b) If you remove opposite corner squares from the chessboard, can you tile the remainder with 31 dominoes?

W2. *Sperner's lemma.* Divide a triangle into lots of baby triangles, so that baby triangles only meet at a common edge or a common vertex. Label each main vertex of the whole triangle by 1, 2, or 3; then label vertices on the (12) side by either 1 or 2, on the (23) side by either 2 or 3, and the (13) side by either 1 or 3. Label the points in the interior in any old way, by any of 1, 2, or 3. Show that any such labeling must contain a baby (123) triangle (in fact, an odd number of them). (Follow-up: how does this generalize to n dimensions? If you know what the Brouwer fixed point theorem is, then try to prove it using Sperner's lemma. This and other fun facts are on Harvey Mudd's "Math Fun Facts" website, `http://www.math.hmc.edu/funfacts/`.)

W3. Suppose 2006 red points and 2006 blue points are given in the plane, no three collinear (and pairwise distinct). Show that one can match up the red points and the blue points so that no two of the corresponding line segments intersect.

1. Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

2. At first, a room is empty. Each minute, either one person enters or two people leave. After exactly 3^{2006} minutes, could the room contain $3^{1000} + 2$ people?

3. If 127 people play in a singles tennis tournament, prove that at the end of the tournament the number of people who have played an odd number of games is even.

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4. Three cities need to be connected to three factories by routes which cannot cross each other. Five villages need to be connected to each other by paths which cannot cross. Prove that both tasks are impossible. Hint: use Euler's formula $V - E + F = 2$.

5. Can a 100×100 square be tiled with 4×1 "quadrominoes"?

6. Three bugs start out at points $(0, 0)$, $(3, 0)$ and $(0, 2)$. They move one at a time, in any order. Each bug can only move in a direction parallel to the line containing the other two bugs. Can two of the bugs switch places while the third ends up where it started? Can they end up at $(1, 1)$, $(6, 2)$, $(3, 4)$?

7. Start with the set $\{3, 4, 12\}$. You are then allowed to replace any two numbers a and b with the new pair $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you transform the set into $\{4, 6, 12\}$?

8. There are three types of piece in a Rubik's cube: centers (which stay essentially fixed, but may rotate in place); edges (there are 12); and corners (there are 8). Each piece has a position (where is it?) and an orientation (is it rotated, flipped or twisted?). Show that it is impossible to swap two edges leaving all other pieces correctly located. (If you are feeling ambitious, you can try the following too: show that it is impossible to swap two corners leaving all other pieces correctly located; and to rotate one face-center by $90°$ leaving all other pieces correctly placed and oriented.

9. *(The "Lights out" game)* Suppose $n \geq 2$ light bulbs are arranged in a row, numbered 1 through n. Under each bulb is a button. Pressing the button will change the state of the bulb above it (from on to off or vice versa), and will also change the neighbors' states. (Most bulbs have two neighbors, but the bulbs on the end have only one.) The bulbs start off randomly (some on and some off). For which n is it guaranteed to be possible that by flipping some switches, you can turn all the bulbs off? (Follow-up: I used this problem on a midterm. Try to guess the course.)

10. A collection of n beetles, each black or white in color, is arranged in a line. On each move, a black beetle turns pink, emitting a chemical which causes its immediate neighbors to switch from black to white, or white to black (as appropriate). Already pink bugs are not affected. Under what starting conditions is it possible for all n bugs to turn pink?

11. The n cards of a deck (where n is an arbitrary positive integer) are labeled $1, 2, \ldots, n$. Starting with the deck in any order, repeat the following operation: if the card on top is labeled k, reverse the order of the first k cards. Prove that eventually the first card will be 1 (so no further changes occur).

12. Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

13. This game is played on the quadrant $m, n \geq 0$ of an infinite chessboard. A bug starts on $(0, 0)$. If there is a bug on square (m, n) and both $(m, n + 1)$ and $(m + 1, n)$ are empty, then the bug replicates, and one clone jumps onto $(m, n + 1)$ and the other clone jumps onto $(m + 1, n)$ leaving (m, n) empty. Find a strategy (or show impossiblity) for evacuating all bugs from the six squares $(0, 0)$, $(0, 1)$, $(1, 0)$, $(0, 2)$, $(1, 1)$, $(2, 0)$.

14. An infinite army of bugs is arrayed on the squares $n \leq 0$ of an infinite chessboard. A bug can move by jumping over an adjacent bug (orthogonally, not diagonally) into an empty square. This kills the jumped-over bug, which evaporates without trace. Can the army send at least one bug to the row $n = 5$?

15. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

E-mail address: `ksound@math.stanford.edu, vakil@math.stanford.edu`