POLYA PROBLEM-SOLVING SEMINAR WEEK 5: INEQUALITIES

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Things to remember. Put everything on the positive side. Sums of squares are non-negative. The arithmetic mean-geometric mean inequality (AM-GM): $\sum_{i=1}^n a_i \geq (\prod a_i)^{1/n}$ if the a_i are non-negative. The triangle inequality (the shortest distance between two points is a straight line). Cauchy's inequality $(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$. Jensen's Inequality. Avoid calculus. Use calculus. Lagrange multipliers.

W1. Suppose a, b, and c are real numbers. Show that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

W2. Prove the AM-GM inequality.

W3. Given distinct positive numbers q, and r, with average p, show that

$$\frac{p^p \cdot p^p}{q^q r^r} < 1.$$

W4. Show that $x > \sin x$ for $x > 0$.

W5. Suppose $x_1, \dots, x_n$ are real numbers in $(0, \pi)$, with average x. Show that

$$\prod_{i=1}^n \left(\frac{\sin x_i}{x_i} \right) \leq \left(\frac{\sin x}{x} \right)^n .$$

1. (a) Find, without using calculus, the minimum surface area of a rectangular box which holds volume V.
(b) Find, without using calculus, the minimum area of an open box — with no top — which holds volume V.

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2. Given a, b, c nonnegative real numbers such that $(a+1)(b+1)(c+1) \leq 8$. Prove that $abc \leq 1$.

3. If $x_1, \ldots, x_n$ are positive real numbers whose product is 1, show that $\sum_{i=1}^n x_i \leq \sum_{i=1}^n x_i{}^2$.

4. The polynomial $4x^4 - ax^3 + bx^2 - cx + 5$ has four positive (real) roots such that $r_1/2 + r_2/4 + r_3/5 + r_4/8 = 1$. Find them.

5. Determine the maximum value of $(\sin A_1)(\sin A_2) \cdots (\sin A_n)$ given that

$$(\tan A_1)(\tan A_2) \cdots (\tan A_n) = 1.$$

(Hint: *don't* use calculus without thinking!)

6. (A number-theoretic inequality) Given integers $0 < a < b < c < d < e$, prove that $1/\operatorname{lcm}(a,b) + 1/\operatorname{lcm}(b,c) + 1/\operatorname{lcm}(c,d) + 1/\operatorname{lcm}(d,e) \leq 15/16$. Generalize to integers $0 < a_1 < a_2 < \cdots < a_k$.

7. For which real numbers c is $\frac{1}{2}(e^x + e^{-x}) \leq e^{cx^2}$ for all real x?

8. *(The Power Mean inequality)* Suppose $a_1, \ldots, a_n$ are positive real numbers. Define the kth power mean as

$$M_k := \left(\frac{a_1^k + \cdots + a_n^k}{n} \right)^{1/k}$$

if $k \neq 0$, and $M_0 = (a_1 \cdots a_n)^{1/n}$. Show that if $a > b$, then $M_a \geq M_b$, with equality if and only if all the a_i's are equal. (This can be very handy! If $k = 1$, we get the arithmetic mean; if $k = 0$ we get the geometric mean; if $k = -1$ we get the *harmonic mean*; if $k = 2$ we get the *quadratic mean*.)

9. Prove the "logarithmic mean" inequality for $a > b > 0$:

$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}.$$

10. Show that for every positive integer n,

$$\left(\frac{2n-1}{e} \right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e} \right)^{\frac{2n+1}{2}}.$$

11. Suppose $f(x)$ is a continuous function $[a,b] \to \mathbb{R}^+$ ($a < b$). Figure out what the integral version of the arithmetic mean should be (see the Power Mean statement above). State the integral version of the quadratic mean-arithmetic mean inequality. Prove it!

(A number of these problems are from Mark Lucianovic: 1(a), 2, 4, 5, 6, and 9.)

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