# POLYA PROBLEM-SOLVING SEMINAR WEEK 7: GENERATING FUNCTIONS AND RECURSIONS

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**The Rules.** These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

**The Hints.** Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

**Warm-up 1.** Give a short formula for $\sum_{i=1}^{n} i\binom{n}{i}$.

**Warm-up 2.** (Vandermonde identity) Prove that for any positive integers $k < m, n$,

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}.$$

**Warm-up 3.** (a) Find a closed-form formula for $A_n$, where $A_0 = 2$, $A_1 = 5$, $A_n = 5A_{n-1} - 6A_{n-2}$. (b) Find a closed-form formula for $B_n$ where $B_0 = 1$, $B_1 = 3$, $B_n = 4B_{n-1} - 4B_{n-2}$.

**Sample recurrence write-up.**

*Problem.* Solve the linearly recurrent equation $f_n = 3f_{n-2} + 2f_{n-3}$, with initial conditions $f_0 = 1$, $f_1 = 1$, $f_2 = 6$.

*Solution.* The characteristic equation for this recurrence is $t^3 - 3t - 2 = 0$, i.e. $(t+1)^2(t-2) = 0$. The solutions are $t = -1$ (with multiplicity 2) and $t = 2$ (with multiplicity 1). Thus the solutions are all of the form $(An + B)(-1)^n + C2^n$. Using the values at $n = 0$, 1, and 2, we see that $A = 1$, $B = 0$, $C = 1$, and that the solution is $n(-1)^n + 2^n$.

**The problems.**

**R1.** Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 \neq 0$ and

$$a_{n+3} = 2a_{n+2} + 5a_{n+1} - 6a_n.$$

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Find all possible values for $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

**R2.** What is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$?

**R3.** Suppose $a_n = -a_{n-1} - a_{n-2}$, and $a_0 = 100$. Find $a_{2004}$. (There doesn't seem to be enough information here!)

**R4.** The sequence $r_1, r_2, \ldots$ satisfies $r_n = (5/2)r_{n-1} - r_{n-2}$, and $r_1 = 2004$. Suppose the sequence converges to a finite real number. Find $r_2$.

**R5.** Find a length two recurrence satisfied by $c_n = \cos n°$.

**R6.** The sequence $G_0, G_1, G_2, \ldots$ consists of every tenth Fibonacci number. Show that there is a linear recursion (e.g. of the form $G_n = aG_{n-1} + bG_{n-2}$) with integer co-efficients.

**R7.** A gambling graduate student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly $n$ points at some time in a sequence of $n$ tosses is $(2 + (-1/2)^n)/3$. (*Hint:* Let $P_n$ denote the probability of scoring exactly $n$ points at some time. Express $P_n$ in terms of $P_{n-1}$, or in terms of $P_{n-1}$ and $P_{n-2}$. Use this linear recursion to give an inductive proof. Even better hint, useful in many circumstances: you've been given the answer, so reverse-engineer the recursion, and then try to prove it.)

**G1.** A standard die is labeled 1, 2, 3, 4, 5, 6. When you roll two standard dice, it is easy to compute the probability of the various sums. For example, the probability of rolling two dice and getting a sum of 2 is 1/36, while the probability of getting a 7 is 1/6. Is it possible to construct a pair of "nonstandard" dice (possibly different from one another) with positive integer labels that nevertheless are indistinguishable from a pair of standard dice, if the sum of the dice is all that matters? (In other words, the probability of rolling a 2 should still be 1/36, etc.)

**G2.** $1/9899 = 0.00\,01\,01\,02\,03\,05\,08\,13\ldots$. (The spaces were added to make the pattern clear.) Explain!

**G3.** *(Important fact!!)* Suppose that in base $p$, $n = n_0 + n_1 p + \cdots + n_k p^k$ and $a = a_0 + a_1 p + \cdots + a_k p^k$. Show that

$$\binom{n}{a} \equiv \prod_{i=1}^{k} \binom{n_i}{a_i} \pmod{p}$$

(Possible application: how many odd binomial coefficients are there in the 2006th row of Pascal's triangle?)

**G4.** Show that for each positive integer $n$, the number of partitions of $n$ into unequal parts is equal to the number of partitions of $n$ into odd parts. For example, if $n = 6$, there are 4 partitions into unequal parts, namely

$$1+5, \quad 1+2+3, \quad 2+4, \quad 6.$$

And there are also 4 partitions into odd parts,

$$1+1+1+1+1+1, \quad 1+1+1+3, \quad 1+5, \quad 3+3.$$

**G5.** Notice that $e^{ax}e^{bx} = e^{(a+b)x}$. Consider both sides as power series. Write down the coefficient of $x^n$ on each side. What equality have you just proved? (Recall that $e^y = \sum_{k=0}^{\infty} y^k/k!$.)

**G6.** For nonnegative integers $n$ and $k$, define $Q(n,k)$ to be the coefficient of $x^k$ in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers $a$ and $b$ with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.) Here's a hint which may help: Note that $(1+x+x^2+x^3)$ factors.

**G7.** A finite sequence $a_1, a_2, \ldots, a_n$ is called $p$-*balanced* if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \cdots$ is the same for any $k = 1, 2, \ldots, p$. Prove that if a sequence with 50 members is $p$-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal to zero.

**G8.** Let $d$ be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

**G9.** Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1 \text{ for } n = 1, 2, 3, \ldots.$$

Prove there exists a real number $a$ such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

**G10.** For $n \geq 1$, let $d_n$ be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n\to\infty} d_n = \infty$.

**G11.** Show that the number of partitions of a positive integer $n$ into parts that are not multiples of three is equal to the number of partitions of $n$ in which there are at most two repeats. For example, if $n = 6$, there are 7 partitions of the first kind, namely

$$1+1+1+1+1+1, \quad 1+1+1+1+2, \quad 1+1+2+2,$$
$$1+1+4, \quad 1+5, \quad 2+2+2, \quad 2+4.$$

There are also 7 partitions of the second kind

$$1+1+4, \quad 1+1+2+2, \quad 1+2+3, \quad 1+5, \quad 2+4, \quad 3+3, \quad 6.$$

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